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ADAPTIVE TRACKING OF A CLASS OF FIRST-ORDER SYSTEMS WITH BINARY-VALUED OBSERVATIONS AND FIXED THRESHOLDS*

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Abstract This paper considers the adaptive tracking problem for a class of first-order systems with binary-valued observations generated via fixed thresholds. A recursive projection algorithm is proposed for parameter estimation based on the statistical properties of the system noise. Then, an adaptive control law is designed via the certainty equivalence principle. By use of the conditional expectations of the innovation and output prediction with respect to the estimates, the closed-loop system is shown to be stable and asymptotically optimal. Meanwhile, the parameter estimate is proved to be both almost surely and mean square convergent, and the convergence rate of the estimation error is also obtained. A numerical example is given to demonstrate the efficiency of the adaptive control law.

Key words Adaptive control, binary-valued observation, optimal tracking, parameter estimation, stochastic system.

1 Introduction

Recently, binary-valued observation systems have attracted a lot of attention[1−9] due to the wide use of binary-valued sensors, such as photoelectric sensors for positions, Hall-effect sensors for speed and acceleration, EGO oxygen sensors in automotive emission control, a one-bit (single-bit) quantizer in analog-to-digital conversion, distributed one-bit wireless sensors, etc.[10]. The controlled output of such systems cannot be measured, and what can be measured and used for designing controller is the information whether the system output is larger than a given scalar, which is called threshold.

The threshold is a key factor for binary-valued output systems, which can be fixed or adjustable. For example, the threshold of oxygen sensors[11] in industry is fixed, which depends on the physical characteristics of sensors and cannot be changed. An example of adjustable threshold is the coding process in communications, which is actually a protocol[12], where the threshold can be set online according to the real need. Compared with the sensor with time-varying thresholds, the one with fixed threshold has lower cost of production because of the less

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requirement of storage space, computational capability and energy, but it has poorer capability of supplying information.

For the binary-valued observation systems, some results on identification, state estimation, and fault detection can be found in [1, 6–7, 10, 13–15]. However, few works appear in literature on adaptive control with set-valued observations. The main reason is that for identification purpose the system input can be assumed to be periodic or normally distributed; while for feedback control, the control input signal is decided by the control targets which may spoil the above assumptions on inputs.

Thus, a control-dependent recursive identification algorithm is strongly demanded and some related results have already been obtained. Zhao and Guo[16] studied the quantization system identification under a class of persistent excitation inputs. Godoya, et al.[3] proposed an iterative batch algorithm for identifying the FIR systems using quantized output data under persistent excitations, and maximum likelihood criterion was achieved as the iterative step goes to infinity. Marelli, You, and Fu[5] investigated the identification of ARMA models with intermittent quantized output observations under the persistently excited inputs and gave an asymptotical optimal adaptive quantization scheme in the sense of minimum estimation error covariance.

Inspired by these works, Guo, Zhang, and Zhao[4] studied the adaptive tracking control of a class of first-order systems with binary-valued observations and time-varying thresholds. Zhao, Guo, and Zhang[9] constructed a two-stage algorithm to discuss the adaptive control of linear systems to track periodic targets with set-valued information. This paper takes a gain system as an example to study the adaptive tracking problem to bounded reference signals via binary-valued observations with fixed thresholds.

Compared with [4], the prominent characteristic of this paper is that the threshold is fixed. As mentioned above, the binary-valued observations with fixed thresholds supply less information than the ones with time-varying thresholds. Thus, the design of the adaptive control laws and the analysis of the closed-loop systems are more difficult. The reference signals in this paper are bounded and more common than the periodic ones in [9]. Accordingly, the adaptive control laws are more complex and the two-stage algorithm in [9] does not work here.

To overcome these difficulties, we will make full use of the statistical property of the system noise to generate an innovation sequence, and then take advantage of the a priori information on the unknown parameter to construct the identification algorithm and design adaptive control laws. It should be pointed out that some techniques developed here can be used to deal with the case with time-varying thresholds. More importantly, the innovation construction method gets rid of some strict constraints on the system inputs, such as periodic and i.i.d. (independent and identically distributed) properties, etc.

For a class of first-order system, this paper proposes a projection algorithm to estimate the unknown parameter, and the adaptive tracking control is constructively designed via the certainty equivalence principle. By use of the conditional expectation of the binary-valued observation with respect to the estimates, it is shown that the closed-loop system is stable and the adaptive control law is asymptotically optimal under some mild a priori information on the unknown parameter, statistical property of the noises and the signal to be tracked. Meanwhile, the convergence of the parameter estimate is proved and its convergent rate is also obtained.

The rest of this paper is organized as follows. Section 2 formulates the problem; Section 3 gives a projection algorithm for parameter estimation and a constructive method of designing adaptive control; Section 4 analyzes the performance of the closed-loop system, including the stability of the closed-loop system and the optimality of the adaptive control; Section 5 uses a numerical example to demonstrate the tracking efficiency; Section 6 gives some concluding remarks and related future works.

2 Problem Formulation

Consider a class of first-order systems of the following form:

$$y_k = \theta u_k + d_k, \quad k = 1, 2, \cdots, \tag{1}$$

where $u_k \in \mathbb{R}$, $\theta \in \mathbb{R}$ and $\{d_k, k \geq 1\}$ are, respectively, the input, unknown parameter and noise; y_k is the controlled output, which cannot be measured but is the target signal to be regulated. The observation that can be measured and used to design control is the following binary-valued signal:

$$s_k = I_{[y_k \leq C]} = \begin{cases} 1, & \text{if } y_k \leq C, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $C \in (-\infty, \infty)$ is a known fixed threshold.

The purpose of this paper is to design an adaptive control to drive the controlled output y_k to track a known reference signal $\{y_k^*\}$. In other words, at time k, we will construct an adaptive control u_k based on the past observations $\{s_1, s_2, \cdots, s_{k-1}, u_1, u_2, \cdots, u_{k-1}\}$ to minimize the following tracking index:

$$J_k = E(y_k - y_k^*)^2. \tag{3}$$

Now, the a priori information about the unknown parameter is given and some conditions about the reference signal and system noise also are presented.

Assumption 1 The a priori information of the unknown parameter θ is that $|\theta| \in [\underline{\theta},\ \overline{\theta}]$, where $\overline{\theta}$ and $\underline{\theta}$ are known constants with $0 < \underline{\theta} < \overline{\theta} < \infty$.

Assumption 2 The target output $\{y_k^*, k \geq 1\}$ is a deterministic signal sequence, and there are known constants $\underline{y}^*$ and $\overline{y}^*$ with $0 < \underline{y}^* \leq \overline{y}^* < \infty$ such that $|y_k^*| \in [\underline{y}^*,\ \overline{y}^*]$.

Assumption 3 $\{d_k, k \geq 1\}$ is an independent and identically distributed (i.i.d.) stochastic sequence with zero mean and finite variance. The distribution function denoted by $F(x)$ of d_1 is assumed to be known. The density function denoted by $f(x)$ of d_1 is symmetrical and monotonically decreasing on $[0, \infty)$ in the sense of $f(x_1) \geq f(x_2)$ with $0 \leq x_1 \leq x_2 < \infty$, and its support contains $[-T, T]$ with $T = |C| + 3\frac{\overline{\theta}\overline{y}^*}{\underline{\theta}}$, i.e., $[-T, T] \subseteq \{x \in (-\infty, \infty) : f(x) \neq 0\}$.

Remark 1 As mentioned in [4], Assumption 1 not only implies that the system is controllable, but also tells us the controllability degree of the system (1). In practice, the choosing of $\overline{\theta}$ mainly depends on the experiential knowledge of θ. And, such choosing is also relatively flexible since the method used in this paper is irrelevant to the exact value of $\overline{\theta}$.

Remark 2 Assumption 2 describes the properties of the reference signals, based on which a control law can be designed to ensure a sufficient persistent excitation condition for parameter estimate. Assumption 3 shows the statistical properties of the system noises with which many kinds of random variables are satisfied, such as normal distribution with zero mean, t-distribution with $n > 2$ degrees of freedom, uniform distribution on $[-U, U]$ with $U \geq T$, and so on.

3 Design of Adaptive Control Law

Firstly, let us consider the case that the parameter θ is known. In this case, the control law that minimizes (3) should satisfy

$$y_k^* = \theta u_k. \tag{4}$$

Substituting the above into (1), we obtain the following closed-loop equation:

$$y_k - y_k^* - d_k = 0,$$

and then

$$J_k = E(y_k - y_k^*)^2 = Ed_k^2 = Ed_1^2.$$

However, in the case that the parameter θ is unknown, we need to estimate it. To do so, we propose the following recursive projection algorithm:

$$\widehat{\theta}_k = \Pi_\Theta\left\{\widehat{\theta}_{k-1} + \beta\frac{P_{k-1}u_k}{1+P_{k-1}u_k^2}\left[F(C-\widehat{\theta}_{k-1}u_k) - s_k\right]\right\}, \tag{5}$$

$$P_k = P_{k-1} - \gamma\frac{P_{k-1}^2u_k^2}{1+P_{k-1}u_k^2}, \tag{6}$$

where $\Theta \triangleq [-\overline{\theta}, \overline{\theta}]$, initial value $|\widehat{\theta}_0| \in [\underline{\theta},\ \overline{\theta}]$ and $P_0 > 0$ can be arbitrarily chosen, $\beta > 0$ and $\gamma \in (0, 1]$ are two real numbers; $\Pi_\Theta(\cdot)$ is a projection operator defined by $\Pi_\Theta(x) = \arg\min_{z\in\Theta}|x-z|$ for any $x \in \mathbb{R}$; $F(\cdot)$ is the distribution function given by Assumption 3; C is the threshold in (2).

According to the certainty equivalence principle, replacing the θ in (4) by its estimate $\widehat{\theta}_{k-1}$, we obtain an equation of the adaptive control law: $y_k^* = \widehat{\theta}_{k-1}u_k$, from which u_k cannot be well defined when $\widehat{\theta}_{k-1} = 0$. Thus, we make the following modification:

$$u_k = \frac{y_k^*}{\widehat{\theta}_{k-1}}I_{[\underline{\theta}\le|\widehat{\theta}_{k-1}|\le\overline{\theta}]} + \frac{y_k^*}{\underline{\theta}}\left(I_{[0<\widehat{\theta}_{k-1}<\underline{\theta}]} - I_{[-\underline{\theta}<\widehat{\theta}_{k-1}\le 0]}\right). \tag{7}$$

The modification form given by (7), different from the conventional one[17], is to make full use of the a priori information about θ.

Figure 1 shows the process of designing the adaptive control law.

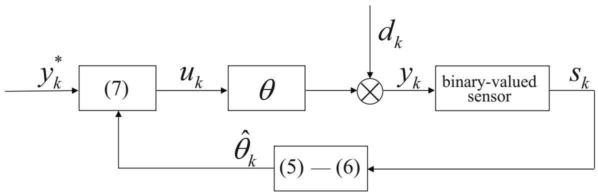

Figure 1 Design mechanism of the adaptive control law

Remark 3 Though the a priori information on θ is $|\theta| \in [\underline{\theta},\ \overline{\theta}]$, we choose $\Theta = [-\overline{\theta}, \overline{\theta}]$ as the projection set since $[-\overline{\theta}, \overline{\theta}]$ is convex and compact, and suitable for projection calculation. The term $F(C-\widehat{\theta}_{k-1}u_k) - s_k$ in (5) can be seen as the innovation from the latest binary-valued observation since $E[s_k|\widehat{\theta}_{k-1}] = F(C-\theta u_k)$. It should be pointed out that the algorithm (5)–(6) would be not convergent if $F(C-\widehat{\theta}_{k-1}u_k) - s_k$ was replaced by $s_k - F(C-\widehat{\theta}_{k-1}u_k)$. It is the very introducing of such kind of innovations that make the adaptive control via set-valued observations and fixed thresholds become possible.

Remark 4 The algorithm (5)–(6) can also work in the case of time-varying thresholds. For example, if C was replaced by $c_k = \widehat{\theta}_{k-1}u_k$, then $2(F(C-\widehat{\theta}_{k-1}u_k) - s_k)$ becomes $I_{[y_k>u_k\widehat{\theta}_{k-1}]} - I_{[y_k\le u_k\widehat{\theta}_{k-1}]}$, which implies the identification algorithm in [4]. But, if c_k was set to be C in [4], then there would be $I_{[y_k>u_k\widehat{\theta}_{k-1}]} - I_{[y_k\le u_k\widehat{\theta}_{k-1}]} = I_{[y_k>C]} - I_{[y_k\le C]}$, from which one cannot get (5)–(6).

4 Stability and Optimality of the Closed-Loop System

In this section, we discuss the performance of the closed-loop system under the control law (7), including the stability and asymptotical optimality. Meanwhile, the convergence of parameter estimates given by the algorithm (5)–(6) is proved and its convergence rate is obtained.

Theorem 1 *Consider the system* (1) *with the binary-valued output* (2) *under the adaptive control* (5)–(7)*. If Assumptions* 1–3 *hold, then the closed-loop system is stable in the sense of* $\sup_{k\geq 0} Ey_k^2 < \infty$.

Proof From (7), it can be seen that $|u_k| \leq \frac{\overline{y}^*}{\underline{\theta}}$. And then, by (1) and Assumption 1, we have

$$|y_k| \leq |\theta||u_k| + |d_k| \leq \frac{\overline{\theta}\overline{y}^*}{\underline{\theta}} + |d_k|.$$

Therefore,

$$y_k^2 \leq \frac{(\overline{\theta}\overline{y}^*)^2}{\underline{\theta}^2} + \frac{2\overline{\theta}\overline{y}^*}{\underline{\theta}}|d_k| + d_k^2,$$

which implies

$$Ey_k^2 \leq \frac{(\overline{\theta}\overline{y}^*)^2}{\underline{\theta}^2} + \frac{2\overline{\theta}\overline{y}^*}{\underline{\theta}}\sqrt{Ed_1^2} + Ed_1^2 < \infty$$

due to Assumption 3 and Schwarz Inequality ([18], pp. 105). Thus, the theorem is proved. ∎

Theorem 2 *Under the conditions of Theorem* 1*, the closed-loop system is asymptotically optimal in the sense of* $\lim_{k\to\infty} J_k = Ed_1^2$*. Meanwhile, the parameter estimates are strongly consistent and mean square convergent to the real parameter:* $\lim_{k\to\infty} \widehat{\theta}_k = \theta$ *a.s. and* $\lim_{k\to\infty} E(\widehat{\theta}_k - \theta)^2 = 0$.

Remark 5 Compared with the solutions provided in [4], some new technical difficulties crop up in the proof of Theorem 2. For example, more detailed and deep analysis about the distribution function of the system noise is needed since $F(\cdot)$ is involved in (5)–(6), and the influence of the fixed threshold C on the closed-loop system has to be carefully considered.

Proof We take three steps to prove the theorem.

Step 1 To prove mean square convergence, i.e., $\lim_{k\to\infty} E(\widehat{\theta}_k - \theta)^2 = 0$.

From (7) it can be seen that

$$u_k \in \mathscr{F}_{k-1} = \sigma(d_i, 1 \leq i \leq k-1) \tag{8}$$

and

$$0 < M_1 \leq |u_k| \leq M_2 < \infty \tag{9}$$

with $M_1 = \frac{\underline{y}^*}{\overline{\theta}}$ and $M_2 = \frac{\overline{y}^*}{\underline{\theta}}$.

Let $\widetilde{\theta}_k = \widehat{\theta}_k - \theta, k = 0, 1, \cdots$, and notice that Θ is a convex-compact set. Then, by the property of the projection operator we have

$$|\widetilde{\theta}_k| \leq \left|\widetilde{\theta}_{k-1} + \beta\frac{P_{k-1}u_k}{1 + P_{k-1}u_k^2}\big[F(C - \widehat{\theta}_{k-1}u_k) - s_k\big]\right|, \tag{10}$$

which together with $(F(C - \widehat{\theta}_{k-1}u_k) - s_k)^2 \leq 1$ implies

$$\widetilde{\theta}_k^2 \leq \widetilde{\theta}_{k-1}^2 - 2\frac{\beta P_{k-1}u_k}{1 + P_{k-1}u_k^2}\widetilde{\theta}_{k-1}\big[s_k - F(C - \widehat{\theta}_{k-1}u_k)\big] + \frac{\beta^2 P_{k-1}^2 u_k^2}{(1 + P_{k-1}u_k^2)^2}.$$

Thus, by (8) we can get

$$\begin{aligned}
&E[\widetilde{\theta}_k^2|\mathscr{F}_{k-1}]\\
&\le \widetilde{\theta}_{k-1}^2+\frac{\beta^2P_{k-1}^2u_k^2}{(1+P_{k-1}u_k^2)^2}-2\frac{\beta P_{k-1}u_k}{1+P_{k-1}u_k^2}\widetilde{\theta}_{k-1}\big(E[s_k|\mathscr{F}_{k-1}]-F(C-\widehat{\theta}_{k-1}u_k)\big)\\
&=\widetilde{\theta}_{k-1}^2+\frac{\beta^2P_{k-1}^2u_k^2}{(1+P_{k-1}u_k^2)^2}-2\frac{\beta P_{k-1}u_k}{1+P_{k-1}u_k^2}\widetilde{\theta}_{k-1}\big(F(C-\theta u_k)-F(C-\widehat{\theta}_{k-1}u_k)\big). \qquad (11)
\end{aligned}$$

Letting $\alpha=2\overline{\theta}M_2$, noticing that $|\widetilde{\theta}_k|\le 2\overline{\theta}$, by (5), (9), and Assumption 1, we know

$$|C-\theta u_k|\le |C|+|\theta||u_k|\le |C|+\overline{\theta}M_2=T-\alpha;$$

and, by (19) in Appendix and $C-\widehat{\theta}_{k-1}u_k=(C-\theta u_k)-u_k\widetilde{\theta}_{k-1}$,

$$u_k\widetilde{\theta}_{k-1}\big(F(C-\theta u_k)-F(C-\widehat{\theta}_{k-1}u_k)\big)\ge f(\zeta)u_k^2\widetilde{\theta}_{k-1}^2, \qquad (12)$$

where $\zeta=\max\limits_{M_1\le|u_k|\le M_2}\{|C-\theta u_k|\}+2\overline{\theta}M_2=|C|+|\theta|M_2+2\overline{\theta}M_2$. Here, we have used the fact that $\max\limits_{M_1\le|u_k|\le M_2}\{|C-\theta u_k|\}=|C|+|\theta|M_2$.

Substituting (12) into (11) results in

$$\begin{aligned}
E[\widetilde{\theta}_k^2|\mathscr{F}_{k-1}]&\le \widetilde{\theta}_{k-1}^2-2\beta f(\zeta)\frac{P_{k-1}u_k^2}{1+P_{k-1}u_k^2}\widetilde{\theta}_{k-1}^2+\frac{\beta^2P_{k-1}^2u_k^2}{(1+P_{k-1}u_k^2)^2}\\
&=\left(1-2\beta f(\zeta)\frac{u_k^2}{u_k^2+P_{k-1}^{-1}}\right)\widetilde{\theta}_{k-1}^2+\frac{\beta^2P_{k-1}^2u_k^2}{(1+P_{k-1}u_k^2)^2}.
\end{aligned}$$

By (9) and (21) in Appendix, we have

$$E[\widetilde{\theta}_k^2|\mathscr{F}_{k-1}]\le\left(1-\frac{2\beta f(\zeta)M_1^2}{M_2^2+P_0^{-1}+\gamma M_2^2(k-1)}\right)\widetilde{\theta}_{k-1}^2+\beta^2M_2^2\left(\frac{1}{P_0}+\frac{\gamma M_1^2(k-1)}{(1-\gamma)P_0M_2^2+1}\right)^{-2},$$

which implies

$$E\widetilde{\theta}_k^2\le\left(1-\frac{2\beta f(\zeta)M_1^2}{M_2^2+P_0^{-1}+\gamma M_2^2(k-1)}\right)E\widetilde{\theta}_{k-1}^2+\beta^2M_2^2\left(\frac{1}{P_0}+\frac{\gamma M_1^2(k-1)}{(1-\gamma)P_0M_2^2+1}\right)^{-2}. \qquad (13)$$

By Assumption 1 and $M_2=\frac{\overline{y}^*}{\underline{\theta}}$, we have

$$\zeta=|C|+|\theta|M_2+2\overline{\theta}M_2\le|C|+\overline{\theta}M_2+2\overline{\theta}M_2=|C|+3\frac{\overline{\theta}\overline{y}^*}{\underline{\theta}}=T. \qquad (14)$$

Since the support of $f(x)$ contains $[-T,T]$, $f(\zeta)>0$. Thus, we can conclude that

$$\frac{\beta^2M_2^2\left(\frac{1}{P_0}+\frac{\gamma M_1^2(k-1)}{(1-\gamma)P_0M_2^2+1}\right)^{-2}}{\frac{2\beta f(\zeta)M_1^2}{M_2^2+P_0^{-1}+\gamma M_2^2(k-1)}}\to 0,\quad \text{as } k\to\infty,$$

which together with (13) and Theorem 1.2.22 of [17] implies $E\widetilde{\theta}_k^2=E(\widehat{\theta}_k-\theta)^2\to 0$, as $k\to\infty$.

Step 2 To prove almost sure convergence, i.e., $\lim_{k\to\infty} \widehat{\theta}_k = \theta$, a.s.

By (11) and (12), we have

$$E[\widetilde{\theta}_k^2|\mathscr{F}_{k-1}] \le \widetilde{\theta}_{k-1}^2 + \frac{\beta^2 P_{k-1}^2 u_k^2}{(1+P_{k-1}u_k^2)^2},$$

and from (21) in Appendix,

$$E\left(\sum_{k=1}^{\infty}\frac{\beta^2 P_{k-1}^2 u_k^2}{1+P_{k-1}u_k^2}\right) < \beta^2 M_2^2 \sum_{k=1}^{\infty}\left(\frac{1}{P_0}+\frac{\gamma M_1^2(k-1)}{(1-\gamma)P_0 M_2^2+1}\right)^{-2} < \infty.$$

Thus, by Lemma 1.2.2 of [19], $|\widetilde{\theta}_k|$ converges almost surely to a bounded limit. Notice that $\lim_{k\to\infty} E\widetilde{\theta}_k^2 = 0$. Then, there is a subsequence of $|\widetilde{\theta}_k|$ that converges almost surely to 0. Consequently, $\widetilde{\theta}_k$ almost surely converges to 0, or equivalently, $\lim_{k\to\infty} \widehat{\theta}_k = \theta$, a.s.

Step 3 To prove asymptotical optimality, i.e., $\lim_{k\to\infty} J_k = Ed_1^2$.

Since $\widehat{\theta}_{k-1} \xrightarrow{\text{a.s.}} \theta$, by (7) we have $\widehat{\theta}_{k-1}u_k - y_k^* \xrightarrow{\text{a.s.}} 0$, and thus,

$$\theta u_k - y_k^* = (\widehat{\theta}_{k-1}u_k - y_k^*) - (\widehat{\theta}_{k-1}-\theta)u_k \xrightarrow{\text{a.s.}} 0. \tag{15}$$

By (9) and Assumptions 1–3, it can be seen that

$$|\theta u_k - y_k^*| \le |\theta u_k| + |y_k^*| \le \frac{\overline{\theta}\overline{y}^*}{\underline{\theta}} + \overline{y}^* < \infty.$$

Thus, by (15) and Lebesgue Dominated Convergence Theorem (see [18], pp. 100), one can get

$$\lim_{k\to\infty} E(\theta u_k - y_k^*)^2 = 0. \tag{16}$$

On the other hand, by (1) we know

$$\begin{aligned} E(y_k - y_k^*)^2 &= E\left(\theta u_k + d_k - y_k^*\right)^2 \\ &= Ed_k^2 + 2E[d_k(\theta u_k - y_k^*)] + E(\theta u_k - y_k^*)^2. \end{aligned}$$

From Assumption 3 and (8), we have $E[d_k(\theta u_k - y_k^*)] = Ed_k E(\theta u_k - y_k^*) = 0$, and thus,

$$E(y_k - y_k^*)^2 = Ed_1^2 + E(\theta u_k - y_k^*)^2.$$

This together with (16) renders

$$J_k = E(y_k - y_k^*)^2 \to Ed_1^2, \quad \text{as} \quad k\to\infty,$$

which implies the asymptotical optimality of the closed-loop system. ∎

Corollary 1 *Under the condition of Theorem 2, the parameter estimate has the following convergence rate*

$$E(\widehat{\theta}_k - \theta)^2 = O\left(\frac{1}{k}\right)$$

with $2\rho f\big(|C| + 3\frac{\overline{\theta}\overline{y}^*}{\underline{\theta}}\big) > \big[(\overline{y}^*\overline{\theta})/(\underline{\theta}\underline{y}^*)\big]^2$ *and* $\rho = \frac{\beta}{\gamma}$.

Proof Since $f(x)$ is monotonically decreasing on $[0,\infty)$, by (13) and (14) we have

$$E\widetilde{\theta}_k^2 \leq \left(1-\frac{2\rho f(|C|+3\overline{\theta}\overline{y}^*/\underline{\theta})(M_1/M_2)^2}{k-1+(M_2^2+P_0^{-1})/\gamma M_2^2}\right)E\widetilde{\theta}_{k-1}^2+\beta^2M_2^2\left(\frac{1}{P_0}+\frac{\gamma M_1^2(k-1)}{(1-\gamma)P_0M_2^2+1}\right)^{-2},$$

which together with Lemma 2 in Appendix implies the corollary. ∎

Remark 6 Corollary 1 describes the influence of β and γ on the convergence rate of the algorithm (5)–(6). Similar to the case of time-varying thresholds, we can choose suitable β and γ such that the convergence rate of the identification algorithm (5)–(6) is of order $\frac{1}{k}$.

5 Simulation

Consider a gain system

$$y_k = \theta u_k + d_k$$

with the binary-valued output observation

$$s_k = I_{[y_k \leq C]}.$$

Here $C = 2$ is the fixed threshold. The time-invariant parameter $\theta = 6$ is unknown, but its range $[1, 15]$ is known, i.e., $\underline{\theta} = 1$, $\overline{\theta} = 15$. The system noise $\{d_k, k \geq 1\}$ satisfies Assumption 3, the density function of d_1 is $f(x) = \frac{1}{\sqrt{0.2\pi}}e^{-50x^2}$ and its distribution function is $F(x) = \int_{-\infty}^{x} f(u)du$.

Our purpose is to make y_k track the reference signal $y_k^* \equiv 16$, which implies $\underline{y}^* = \overline{y}^* = 16$.

According to (7), we have the following adaptive control law:

$$u_k = \frac{16}{\widehat{\theta}_{k-1}}I_{[1\leq|\widehat{\theta}_{k-1}|\leq 15]} + \frac{16}{1}\left(I_{[0<\widehat{\theta}_{k-1}<1]} - I_{[-1<\widehat{\theta}_{k-1}\leq 0]}\right), \tag{17}$$

where $\widehat{\theta}_{k-1}$ is given by (5)–(6) with $\beta = 1$ and $\gamma = 0.5$ and initial values $\widehat{\theta}_0 = P_0 = 1$.

Figure 2 describes a trajectory of y_k under the control (17) within 1000 steps. The system output y_k seems to be a white noise around the target y_k^*, which means that θu_k has tracked the target and the error is only caused by the system noise d_k.

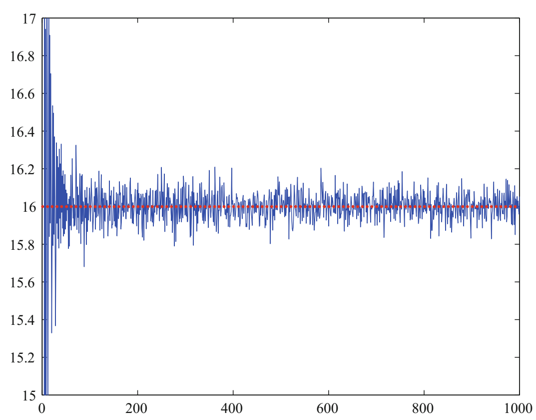

Figure 2 Tracking performance of y_k (solid) to $y^* \equiv 16$ (dashed)

Figure 3 shows the estimation convergence of $\widehat{\theta}_k$ in trajectory, which and the performance in Figure 2 are both consistent with the results of Theorem 2.

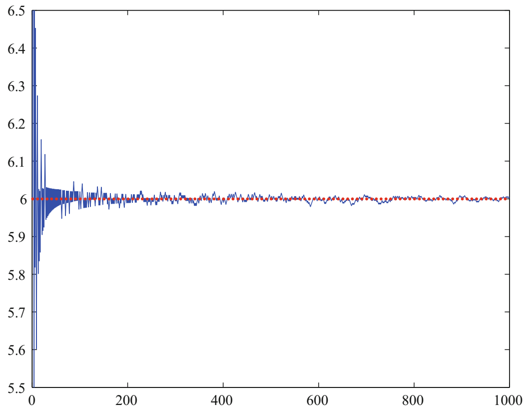

Figure 3 Estimation convergence of $\widehat{\theta}_k$ (solid) to the true parameter $\theta = 6$ (dashed)

6 Conclusion

In this paper, we have studied the adaptive tracking control via binary-valued observations with fixed threshold. An innovation sequence in the identification algorithm was introduced. Under some mild conditions on the a priori knowledge of the unknown parameters and reference signals, we proved the stability of the closed-loop system and the asymptotical optimality of the adaptive tracking, and obtained a convergence rate of identification algorithms.

There are many challenging and meaningful open problems in this fields, such as how to design identification and adaptive control laws with multi-threshold quantization observations, how to deal with the more general cases of system models, etc. The idea of introducing a proper kind of innovation sequence into the identification algorithm may help us to solve these general identification and adaptive problems.

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Appendix

Lemma 1 *Assume that α and b are real numbers with $0 < \alpha < T$ and $\alpha + |b| \le T$. Then, for the functions $F(\cdot)$ and $f(\cdot)$ defined in Assumption 3, the following inequality holds for any $x \in [-\alpha, \alpha]$,*

$$x\big(F(b) - F(b-x)\big) \ge f(|b| + \alpha)x^2. \tag{18}$$

Furthermore, if $b = b(x) \in [\alpha - T, T - \alpha]$ is a continuous function of x, then for any $x \in [-\alpha, \alpha]$,

$$x\big(F(b(x)) - F(b(x) - x)\big) \ge f(B + \alpha)x^2 \tag{19}$$

with $B = \max\limits_{x \in [-\alpha, \alpha]} |b(x)|$.

Proof Since the support of $f(x)$ contains $[-T, T]$, $F(x)$ is continuously differentiable on $[-T, T]$. By the differential mean value theorem and $\alpha + |b| \le T$, for any $x \in [-\alpha, \alpha]$, there exists $\xi = \xi(x)$ between b and $b - x$ such that $F(b) - F(b - x) = f(\xi)x$. Noticing $f(x)$ is symmetrical and monotonically decreasing on $[0, \infty)$, we have

$$x\left(F(b) - F(b-x)\right) = f(\xi)x^2 \ge f\left(\max\{|b - \alpha|, |b + \alpha|\}\right)x^2,$$

which implies (18) by $\max\{|b-\alpha|, |b+\alpha|\} = |b| + \alpha$.

Now, we prove (19). Noticing that $f(x)$ is monotonically decreasing on $[0, \infty)$, for any $x \in [0, \alpha]$, by (18) we have

$$\begin{aligned} x\big(F(b(x)) - F(b(x)-x)\big) &\geq x\big(F(-B) - F(-B-x)\big) \\ &\geq f\big(|-B| + \alpha\big)x^2 = f(B+\alpha)x^2. \end{aligned}$$

Similarly, for any $x \in [-\alpha, 0)$, we have

$$\begin{aligned} x\big(F(b(x)) - F(b(x)-x)\big) &= -x\big(F(b(x)-x) - F(b(x))\big) \\ &\geq -x\big(F(B-x) - F(B)\big) = x\big(F(B) - F(B-x)\big) \\ &\geq f\big(B+\alpha\big)x^2. \end{aligned}$$

Thus, (19) is true. ∎

Lemma 2[4] *Suppose that $\{x_k, k \geq 1\}$ is a sequence of real numbers such that for all sufficiently large k,*

$$x_k \leq \Big(1 - \frac{\lambda}{k+a}\Big)x_{k-1} + \frac{\mu}{(k-1)^{2+\delta}},$$

where $a \in \{x : x \in \mathbb{R}, x \neq -1, -2, \cdots\}$, $\lambda > 0$, $\delta \geq 0$. Then

$$x_k = \begin{cases} O\Big(\dfrac{1}{k^\lambda}\Big), & 0 < \lambda < 1+\delta, \\ O\Big(\dfrac{\log k}{k^{1+\delta}}\Big), & \lambda = 1+\delta, \\ O\Big(\dfrac{1}{k^{1+\delta}}\Big), & \lambda > 1+\delta. \end{cases}$$

Lemma 3[4] *Assume there exist constants $M_2 > M_1 > 0$ such that $M_1 \leq |u_k| \leq M_2$. Then, for any initial value $P_0 > 0$, P_k have the following properties:*

$$0 < P_{k+1} < P_k \quad \textit{and} \quad \lim_{k\to\infty} P_k = 0; \tag{20}$$

$$\Big(\frac{1}{P_0} + \gamma M_2^2 k\Big)^{-1} \leq P_k \leq \Big(\frac{1}{P_0} + \frac{\gamma M_1^2}{(1-\gamma)P_0M_2^2 + 1}k\Big)^{-1}. \tag{21}$$